Optimal Variable-Thrust Transfer of a Power-Limited Rocket between Neighboring Circular Orbits

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Minimum-fuel transfers between neighboring circular orbits for low-thrust propulsion systems have been studied The analysis is based on the simplification that only small deviations from an original circular orbit are allowed, so that the gravitational terms in the equations of motion may be linearized A completely analytical solution is determined, and the resultant thrust-vector control programs are compared with those obtained previously by Hinz for constant-thrust acceleration

Nomenclature

a = thrust to mass ratio
$\bar{a}$ = average thrust acceleration defined by Eq (55)
A = a/ω_0
C = constant of integration
f = rate of change of a state variable
F = fundamental function
i = inclination of orbit planes
J = defined by Eq (1)
J_1 = defined by Eq (50)
J_2 = defined by Eq (51)
n = 0, 1, 2,
r = radius
t = time
u, v, w = velocity components
x, y, z = position coordinates in spherical system
x', y', z' = position coordinates in rectangular system
λ = Lagrange multiplier
τ = $\omega_0 t$
ω = angular velocity in circular orbit

Subscripts

f = final condition
i = index denoting $x, y\ z, u, v, w$
j = index denoting $x, y\ z$
0 = initial condition

Introduction

IT is characteristic of high-specific-impulse, low-thrust propulsion systems that the source of power is separate from the thrust device itself Consequently, such propulsion systems are referred to as power limited, since thrust is restricted in magnitude by the output of the power supply, which is in turn limited by the necessity of minimizing power supply weight

The problem of transferring between neighboring circular orbits by a power-limited rocket is of interest for two basic reasons First of all, the problem can be solved analytically, provided that the thrust acceleration is not constrained in magnitude and that the proper simplifying assumptions are made in the mathematical model of the system The analytic expressions thus obtained for the controls and for the optimum trajectories then provide insight into more general transfer problems where the simplifying restrictions are lifted Secondly, the solution to this problem provides a lower bound to the performance requirements for circle-to-circle transfers

It is interesting to note that if, for the same system model as has been used herein, the thrust acceleration is assumed constant,[1] analytic integration of the equations of motion does not appear feasible Therefore, allowance for variable-thrust acceleration is essential if completely analytic solutions are to be obtained

Analytical Method

Description of the Mathematical Model

The phrase "neighboring circular orbits," as defined here, requires that the inclination between orbit planes be small and that the radial separation between orbits be small relative to the radius of either circle If it is further assumed that motion in the transfer orbit does not deviate significantly from these neighboring circles, linearization of the equations of motion is permissible

The choice between a rotating rectangular or spherical coordinate system in this problem is somewhat arbitrary in that, upon linearization, equivalent equations are obtained for the two cases It is significant, however, that the linearizing assumptions made are slightly different in each of these coordinate systems, and an important conclusion can be drawn from this feature

Consider the coordinate system depicted in Fig 1, a rectangular system with its origin fixed on the interior orbit (assumed to be the reference orbit) in the x', y' plane The mutually orthogonal coordinates x', y', z' form a triad that revolves with the angular speed ω_0 of the reference orbit, so that motion in this frame of reference is relative to a point on the reference orbit The spherical coordinate system in Fig 2 is described by the arc x in the plane of the reference orbit, the arc z measured normal to this plane, and a radial dimension y

In order to linearize the equations of motion in the first system, it is necessary to assume that excursions x', y', z' from the origin be small in comparison with the radius r_0 of the reference orbit Motion is therefore constrained to a small sphere about the origin No restrictions are placed on the component velocities In the spherical system, only the assumption of small component velocities will linearize the equations, whereas the arc x is not limited The resultant motion is constrained to a torus about the reference orbit

Since the linearized equations of motion are identical except for differences in notation,[2] one can draw the conclusion that, if in the spherical system the resultant motion does not involve large variations in x, the velocity components may be large In the present study the spherical system has been used throughout, and the results may be extended according to the foregoing discussion

Analysis

The optimization problem is to derive the optimal control equations for the minimum-fuel transfer of a power-limited rocket with unbounded thrust acceleration between neigh

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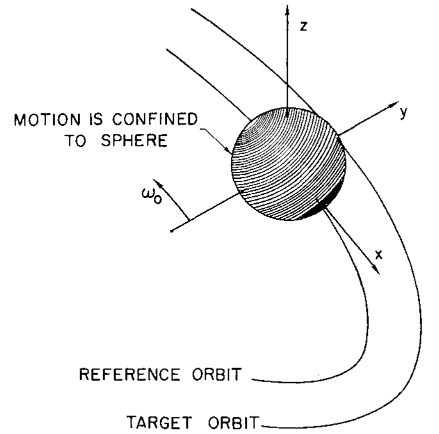

Fig 1 Rectangulaı cooıdinate system

boring circular orbits in a given time Mathematically, this requires minimization of the integral

$$J = \int_0^{t_f} \frac{a^2}{2}\, dt \tag{1}$$

subject to constraints imposed by the equations of motion The controls that are to be optimized are the components of the thrust acceleration vector **a**

The problem may be treated as a problem of Lagrange in the calculus of variations In particular, Breakwell's formulation[3] of this problem is used because the linearized equations in the present case are particularly suited to this formulation The basic equations of the variational problem are

$$J = -\int_0^{\tau_f} \frac{\omega_0 A^2}{2}\, d\tau \tag{1}$$

$$f_0 = -\frac{\omega_0 A^2}{2} = -\frac{\omega_0}{2}(A_x^2 + A_y^2 + A\ ^2) \tag{2}$$

$$f_1 = dx/d\tau = u \tag{3}$$

$$f_2 = dy/d\tau = v \tag{4}$$

$$f_3 = dz/d\tau = w \tag{5}$$

$$f_4 = du/d\tau = A_x + 2v \tag{6}$$

$$f_5 = dv/d\tau = A_y + 3y - 2u \tag{7}$$

$$f_6 = dw/d\tau = A\ - z \tag{8}$$

where Eqs (6–8) are the linearized equations of motion, and the nondimensional time parameter τ is used throughout Note that Eqs (5) and (8), representing the "out-of-plane" motion, are independent of the "in-plane" components This independence indicates that out-of-plane motion is uncoupled from planar motion in this linearized problem

These equations, together with the Lagrange multipliers λ_i, are used to form the fundamental function:

$$F = -(\omega_0/2)(A_x^2 + A_y^2 + A\ ^2) + \lambda_x u + \lambda_y v + \lambda\ w + \lambda_u(A_x + 2v) + \lambda\ (A_y + 3y - 2u) + \lambda_w(A\ - z) \tag{9}$$

The Euler-Lagrange equations defined by Eq (10) comprise necessary conditions for the existence of an extremal arc:

$$d\lambda_i/d\tau = -\partial F/\partial x_i \tag{10}$$

where the subscripts i refer to the state variables x, y, z, u, v, and w

For the control variables A_x, A_y, and A , these equations take the form

$$\partial F/\partial A_j = 0 \tag{11}$$

The Euler-Lagrange equations are summarized in Eqs (12–20), where the dot denotes differentiation with respect to τ:

$$\dot\lambda_x = 0 \tag{12}$$

$$\dot\lambda_y = -3\lambda \tag{13}$$

$$\dot\lambda = \lambda \tag{14}$$

$$\dot\lambda_u = -\lambda_x + 2\lambda \tag{15}$$

$$\dot\lambda = -\lambda_y - 2\lambda_u \tag{16}$$

$$\dot\lambda_w = -\lambda \tag{17}$$

$$\lambda_u = \omega_0 A_x \tag{18}$$

$$\lambda = \omega_0 A_y \tag{19}$$

$$\lambda_w = \omega_0 A \tag{20}$$

Integration of these equations introduces six unknown constants:

$$\lambda_x = \omega_0 C_0 \tag{21}$$

$$\lambda_y = -6\omega_0(C_4 + C_0\tau - C_1 \cos\tau + C_2 \sin\tau) \tag{22}$$

$$\lambda = 2\omega_0(C_5 \sin\tau + C_3 \cos\tau) \tag{23}$$

$$\lambda_u = \omega_0(3C_4 + 3C_0\tau - 4C_1 \cos\tau + 4C_2 \sin\tau) \tag{24}$$

$$\lambda = 2\omega_0(C_0 + C_1 \sin\tau + C_2 \cos\tau) \tag{25}$$

$$\lambda_w = 2\omega_0(C_5 \cos\tau - C_3 \sin\tau) \tag{26}$$

Before proceeding to integration of the equations of motion, it is convenient here to investigate boundary conditions and transversality conditions for this problem All initial conditions are zero, since the starting point is at point O in Fig 2, and all motion in the spherical system is relative to point O Thus

$$x_0 = y_0 = z_0 = u_0 = v_0 = w_0 = 0 \tag{27}$$

The linearized end conditions at the final time τ_f were shown in Refs 1 and 4 to be

$$u_f = \tfrac{3}{2} y_f \qquad v_f = 0 \qquad w_f = (r_0^2 i^2 - z_f^2)^{1/2} \tag{28}$$

where the end position coordinates x_f and z_f are not specified Since these coordinates are free, transversality conditions at

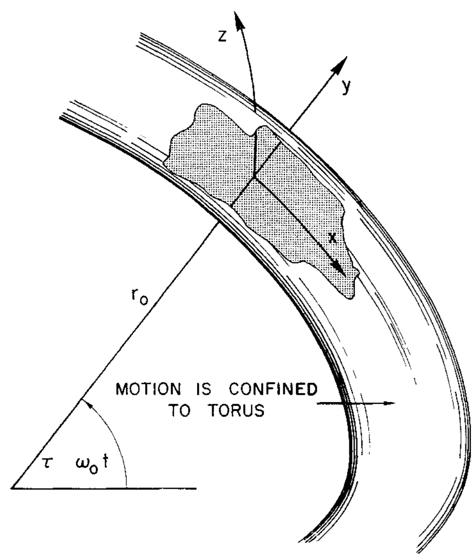

Fig 2 Spherical cooıdinate system

$\tau = \tau_f$ require that

$$\lambda_x^f = \lambda^f = 0 \tag{29}$$

But λ_x is a constant from Eq (21), so that

$$\lambda_x = C_0 = 0 \tag{30}$$

Condition (29) on λ yields the result

$$C_5 = -(C_3/\tan\tau_f) \tag{31}$$

When Eqs (18–31) are substituted into the equations of motion (3–8), the following expressions are obtained for the state variables as functions of τ:

$$x = [16(1 - \cos\tau) - 10\tau \sin\tau]C_1 + [22 \sin\tau - 10\tau \cos\tau - 12\tau]C_2 - [\tfrac{9}{2}\tau^2 - 12(1 - \cos\tau)]C_4 \tag{32}$$

$$y = [5(\sin\tau - \tau \cos\tau)]C_1 + [5\tau \sin\tau - 8(1 - \cos\tau)]C_2 + [6 (\sin\tau - \tau)]C_4 \tag{33}$$

$$z = [\tau \cos\tau - \sin\tau - (\tau \sin\tau/\tan\tau_f)]C_3 \tag{34}$$

$$u = [6 \sin\tau - 10\tau \cos\tau]C_1 + [10\tau \sin\tau - 12(1 - \cos\tau)]C_2 + [12 \sin\tau - 9\tau]C_4 \tag{35}$$

$$v = [5\tau \sin\tau]C_1 + [5\tau \cos\tau - 3 \sin\tau]C_2 - [3(1 - \cos\tau)]C_4 \tag{36}$$

$$w = -\{\tau \sin\tau + [(\sin\tau + \tau \cos\tau)/\tan\tau_f]\}C_3 \tag{37}$$

where the remaining constants C_1, C_2, C_3, and C_4 are functions of the final time τ_f and the known quantities r_0, i, and y_f:

$$C_1 = \frac{y_f \sin\tau_f}{16(1 - \cos\tau_f) - \tau_f(5\tau_f + 3 \sin\tau_f)} \tag{38}$$

$$C_2 = \frac{-y_f(1 - \cos\tau_f)}{16(1 - \cos\tau_f) - \tau_f(5\tau_f + 3 \sin\tau_f)} \tag{39}$$

$$C_3 = \frac{r_0 i \sin\tau_f}{(\tau_f^2 + \tau_f \sin 2\tau_f + \sin^2\tau_f)^{1/2}} \tag{40}$$

$$C_4 = \frac{(y_f/6)(5\tau_f + 3 \sin\tau_f)}{16(1 - \cos\tau_f) - \tau_f(5\tau_f + 3 \sin\tau_f)} \tag{41}$$

The optimal controls can also be obtained as functions of τ from Eqs (18–26 and 38–41):

$$A_x = 3C_4 - 4C_1 \cos\tau + 4C_2 \sin\tau \tag{42}$$

$$A_y = 2(C_1 \sin\tau + C_2 \cos\tau) \tag{43}$$

$$A = -2C_3[\cos(\tau_f - \tau)/\sin\tau_f] \tag{44}$$

The minimized parameter J may be written as

$$\frac{J}{\omega_0^3 r_0^2} = \frac{(y_f/r_0)^2(5\tau_f + 3 \sin\tau_f)}{8[\tau_f(5\tau_f + 3 \sin\tau_f) - 16(1 - \cos\tau_f)]} + \frac{i^2(2\tau_f + \sin 2\tau_f)}{2(\tau_f^2 + \tau_f \sin 2\tau_f + \sin^2\tau_f)} \tag{45}$$

Thus far the Euler-Lagrange equations are the only necessary conditions that have been satisfied by the optimum trajectory The Pontryagin maximum principle provides additional necessary conditions Note that in Eq (9) the fundamental function F and its derivatives are continuous with respect to the controls A_x, A_y, A , so that an extremum can occur either at the stationary point or on the boundaries of the control space The Legendre condition yields the result

$$\frac{\partial^2 F}{\partial A_x^2} = \frac{\partial^2 F}{\partial A_y^2} = \frac{\partial^2 F}{\partial A^2} = -\omega_0 < 0 \tag{46}$$

which is sufficient to insure a local maximum at the stationary point, since the cross second partials are all zero At the boundaries of the control space

$$|A\ | = |A_y| = |A\ | = \infty \tag{47}$$

and the function F is always less than at the stationary point This result satisfies the maximum principle and indicates that the stationary point is an absolute maximum Although the trajectories investigated in this report satisfy two necessary conditions, it should be emphasized that these two conditions are not sufficient for a minimum of J A sufficiency proof would require consideration of the additional necessary condition due to Jacobi

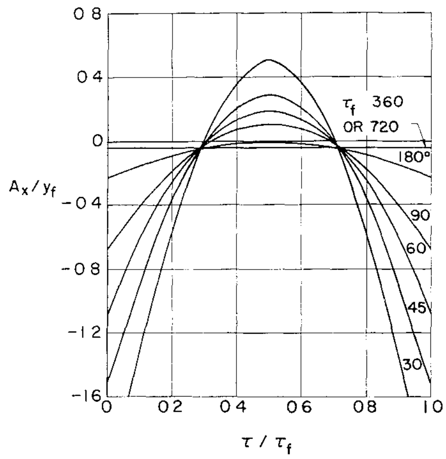

Fig 3 Circumferential component of acceleration

Results

Summary curves for the components of the optimal thrust acceleration control vector are shown in Figs 3–5 In each figure the control is plotted against normalized time τ/τ_f for a range of trip times covering at least one orbital period of the reference orbit ($\tau_f = 2\pi$)

The in-plane components A_x/y_f and A_y/y_f are seen to display symmetry about the midpoint in time for all trip times, whereas the out-of-plane component $A\ /r_0 i$ is symmetrical only for trip times of $2n\pi$ or $(2n + 1)\pi$ In particular, when $\tau_f = 2n\pi$, the components A_x/y_f and A_y/y_f are constant with time, and the latter is zero For the coplanar problem, constant circumferential thrust acceleration is thereby specified as the optimum mode for integer multiples of the period

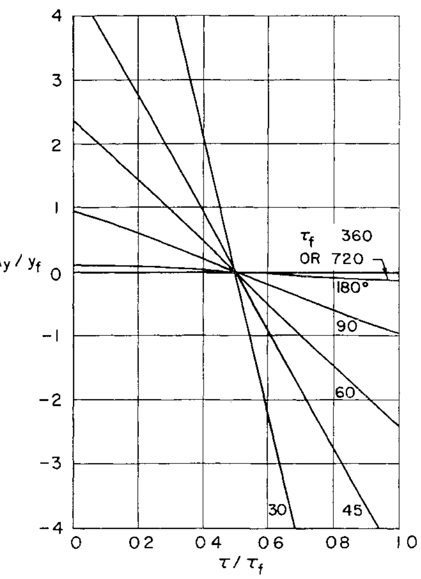

Fig 4 Radial component of acceleration

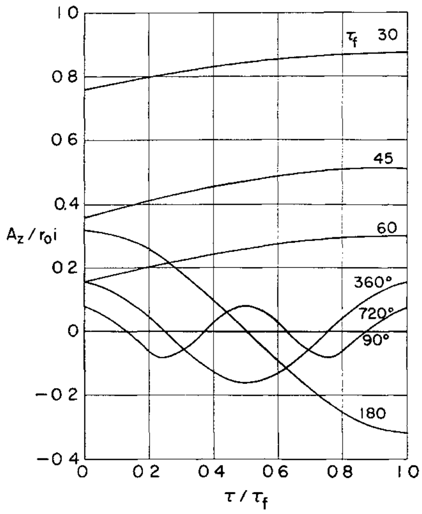

Fig 5 Normal component of acceleration

of the reference orbit, a result that is in agreement with Ref 1

It was explained previously that, in the orbit transfer problem, final values of x and z are not specified, i e , these end conditions are free Using Eqs (32, 34, and 38–41), the optimal values resulting from the variational analysis are found to be

$$x_f/y_f = \tfrac{3}{4}\tau_f \tag{48}$$

and

$$\frac{z_f}{r_0} = \frac{i \sin^2\tau_f}{2(\tau_f^2 + \tau_f \sin 2\tau_f + \sin^2\tau_f)^{1/2}} \tag{49}$$

The first of these equations expresses a result that was not expected to be so simple even in this, a linearized problem Together with the first of Eqs (28), Eq (48) requires that, if rendezvous with a body in the outer orbit is desired, the body must be at $x = 0$ when $\tau = \tau_f/2$, i e , the body from which departure is made and the target body are in conjunction at $\tau = \tau_f/2$ The second equation indicates that, for trip times that are multiples of π, $z_f = 0$, corresponding to transfers that end at a node

Another result that is of interest is the behavior of the minimum J with transfer time In order to demonstrate this behavior, the function J in Eq (45) has been rewritten as the sum of the two terms J_1 and J_2, which are defined in Eqs (50) and (51):

$$\frac{J_1}{(y_f/r_0)^2} = \frac{5\tau_f + 3\sin\tau_f}{8[\tau_f(5\tau_f + 3\sin\tau_f) - 16(1 - \cos\tau_f)]} \tag{50}$$

$$\frac{J_2}{i^2} = \frac{2\tau_f + \sin 2\tau_f}{2[\tau_f^2 + \tau_f \sin 2\tau_f + \sin^2\tau_f]} \tag{51}$$

The term $J_1/(y_f/r_0)^2$ represents the in-plane contribution, and J_2/i^2 is the out-of-plane contribution Since both of these terms are functions of τ_f only, two curves suffice to express the dependence of J on τ_f, and these have been plotted in Fig 6 Both curves display a damped oscillation with transfer time, diverging as τ_f approaches zero and smoothing out as τ_f becomes large

For very short trip times the magnitude of the thrust acceleration must be large, even large enough to completely overshadow the gravitational forces Thus the physical situation becomes analogous to a coplanar transfer between two positions in field-free space This simplified problem has been treated in Ref 5 for variable-thrust acceleration and in Ref 6 for constant acceleration In Ref 5 the magnitude of J is found to be

$$J = 6\omega_0 y_f^2/\tau_f^3 \tag{52}$$

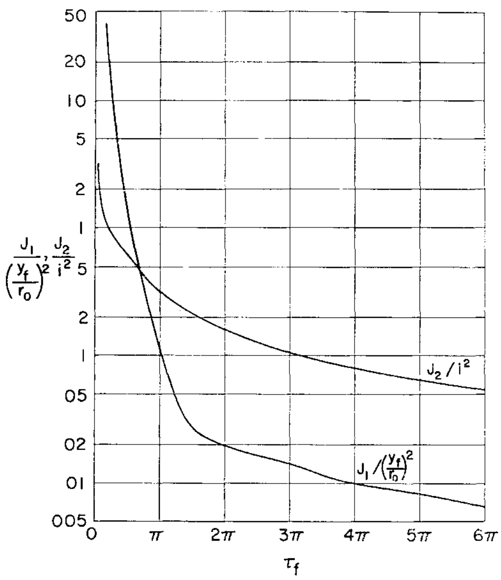

Fig 6 Behavior of components of J with transfer time

a result that can be obtained from Eq (45) with τ_f small and $i = 0$ For $i \neq 0$, the general equation for small τ_f is

$$\frac{J}{\omega_0 r_0^2} = \frac{1}{2\tau_f}\left[\frac{12}{\tau_f^2}\left(\frac{y_f}{r_0}\right)^2 + i^2\right] \tag{53}$$

The first term in the brackets in Eq (53) represents the in plane component and the second the out-of-plane component of J It can be shown that the first term corresponds to changing position only, whereas the second term corresponds to changing velocity only

For very long transfer times, Eq (45) becomes

$$\frac{J}{\omega_0 r_0^2} = \frac{1}{8\tau_f}\left[\left(\frac{y_f}{r_0}\right)^2 + 8i^2\right] \tag{54}$$

and the relative contributions of the in-plane and out-of plane components are no longer functions of τ_f This result was obtained previously in Ref 7 by assuming small changes in the orbital elements over each revolution and summing over many revolutions

One further performance comparison is of interest In Ref 1 an "altitude gain" parameter $y_f\omega_0^2/a\tau_f$ was defined to illustrate the increase in circular orbit radius which can be obtained using constant-thrust acceleration in the planar problem In order to compare the present results with those of Ref 1, an average-thrust acceleration

$$\bar{a} = \left[\int_0^{\tau_f} a^2 d\tau \Big/ \int_0^{\tau_f} d\tau\right]^{1/2} \tag{55}$$

was defined for the variable-thrust transfers This is the

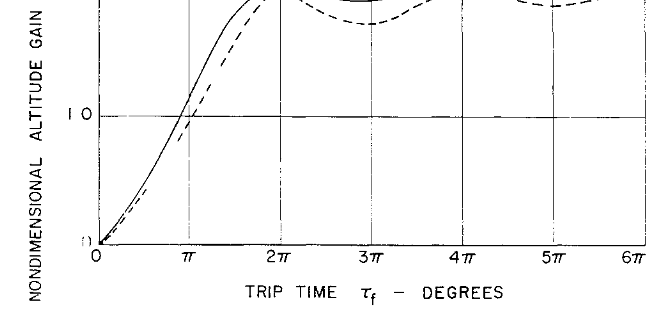

Fig 7 Variation of altitude gain with trip time for coplanar circle-to-circle transfer (———variable thrust, — — —constant thrust)

constant average value of a required to use the same amount of fuel in the same transfer time as would a constant a program For transfer between given coplanar circular orbits in a specified time, $\bar{a}$ is always less than or equal to the required constant a to accomplish the same transfer The altitude gain parameter is plotted in Fig 7 for both modes, constant- and variable-thrust acceleration As anticipated, the variable acceleration curve yields superior performance for all transfer times except integer multiples of the orbital period of the reference orbit At these times, constant-thrust acceleration is optimal, and the curves are tangent to one another

Conclusions

1) The coplanar solutions for values of transfer time equal to an integral multiple of the orbital period are found to reduce to constant circumferentially directed acceleration

2) At other values of transfer time, the optimum thrust magnitude program is shown to yield better performance than the constant-acceleration program

3) The optimal change in angular circumferential position is a linear function of transfer time

4) For all but short transfer times, the restriction on the velocity components imposed by linearization can be lifted

References

[1] Hinz, H K, "Optimal low thrust near circular orbit transfer," AIAA J **1**, 1367–1371 (1963)

[2] Kelley, H J and Dunn, J C, "An optimal guidance approximation for quasi-circular orbital rendezvous," Second IFAC Congress on Automatic Control, Basle, Switzerland (September 1963)

[3] Breakwell, J V, "The optimization of trajectories," J Soc Ind Appl Math **7**, 215–222 (June 1959)

[4] Clohessy, W H and Wiltshire, R S "Terminal guidance system for satellite rendezvous," J Aerospace Sci **27**, 653–658 (1960)

[5] Leitmann, G, "Minimum transfer time for a power limited rocket," J Appl Mech **28**, 171–178 (1961)

[6] Preston, T H, "Two aspects of the time element in interplanetary flight," *Bericht uber den V Internationalen Astronautischen Kongress* (1955), pp 150–156

[7] Edelbaum, T N, "Theory of maxima and minima," *Optimization Techniques*, edited by G Leitmann (Academic Press Inc, New York, 1962), Chap 1

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Surface Strains in Case-Bonded Models of Rocket Motors

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Strain measurements on the case of rocket engine models with applied loads were made with electric-resistance-type strain gages, enabling the computation of case surface stresses The model consisted of an aluminum case with a bonded, viscoelastic grain A model having a circular-shaped grain was tested first with an applied internal pressure and then was tested to failure with an applied axial load, uniformly distributed A model having a star-shaped grain was loaded with an applied internal pressure until a failure by rupturing occurred

Introduction

THE reported tests were undertaken in order to quantitatively ascertain the allowable loads on model rocket engines tested in an extensive, exploratory program in which the internal displacements in solid propellant grains were experimentally determined by observing embedded particles with an x-ray scintillation detection facility [1–3] Models cast from inert and live grains but without cases, had been successfully tested previously using the x-ray facility The accuracy of the x-ray particle detection system was thoroughly investigated in another study [4]

The model rocket engines consisted of an aluminum case and a viscoelastic grain The grain material was an inert polyurethane that had mechanical properties similar in nature to some of the commonly used solid propellants

One of the models had a circular-shaped grain and was tested first with an applied internal pressure and then was tested to failure with an applied axial load, uniformly distributed The other model had a star-shaped grain and was tested with an applied internal pressure until a failure of the grain by rupture occurred

The particular type of end restraints imposed on the models consisted of a platen arrangement whereby the ends of the models were free to translate longitudinally but were restrained in the radial direction Electrical-resistance-type strain gages bonded to the outer surface of the cases were used for measuring the applied strains, from which the case stresses were calculated The gages were metal film, type C6 111§ having a $\frac{1}{16}$-in gage length, a 2 01 ± 1% gage factor, and a 120 ± 0 5-ohm resistance Dummy gages, mounted on a sample of the same type of material from which the model cases were fabricated, were used for temperature compensation, since the C6 gage is self-temperature compensated for use only on steel The strain gage installations are shown in Fig 1

Experimental Procedure

Internal Pneumatic Pressure Tests of Model Having a Circular-Shaped Grain

The model was subjected to internal pneumatic pressures of 0 to 160 psig in 20-psig increments The Baldwin-Tate

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